Darboux Transformation, Lax Pairs, Exact Solutions of Nonlinear Schrödinger Equations, and Soliton Molecules

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General Outline

Lecture I: Darboux Transformation (DT) and Lax Pair

- Linear and Nonlinear Partial Differential Equations

Lecture II: Lax Pair Search Method and Integrability

Lecture III: Single soliton solutions

- Vibrating harmonic trap
- Optical Lattice
- Rogue waves

Lecture IV: Two solitons solutions (Soliton Molecules)

- Instructive Form of the Exact solution
- Formation, Stability, and Dynamics of Soliton Molecules
- Methods of Stabilizing Soliton Molecules
Lecture I
Darboux Transformation (DT)
and Lax Pair

Outline

I Darboux Transformation (DT)

II DT for Linear Ordinary Differential Equations
   - Schrödinger Equation with a Harmonic Potential

III DT for Nonlinear Partial Differential Equations
   - The Korteweg-de-Vries (KdV) Equation
   - The Zakharov-Shabat Method
   - The Nonlinear Schrödinger (NS) Equation

IV Search for the Lax Pairs

V Exact Solitonic Solutions of the Gross-Pitaevskii Equation
**Darboux Transformation**

Consider a Sturm-Liouville equation

\[-\Psi_{xx}(x) + u(x)\Psi = \lambda \Psi(x) ,\]

(1)

with known fixed eigensolution \(\Psi_1\) and eigenvalue \(\lambda_1\).

The Darboux transformation on \(\Psi\) is defined as:

\[
\Psi[1] = \left( \frac{d}{dx} - \frac{\Psi_{1x}}{\Psi_1} \right) \Psi .
\]

(2)

**Theorem:** The function \(\Psi[1]\) satisfies

\[-\Psi[1]_{xx} + u[1]\Psi[1] = \lambda \Psi[1] ,\]

(3)

with

\[u[1] = u - 2 \left( \frac{\Psi_{1x}}{\Psi_1} \right)_x .\]

(4)

Eq.(1) is said to be *covariant* under the DT.
DT for Linear Ordinary Differential Equations

- Schrödinger Equation With a Harmonic Potential

Consider the equation

$$\Psi_{xx} + x^2 \Psi = \lambda \Psi,$$

with seed solution \( \Psi_1 = \exp \left( \frac{x^2}{2} \right) \) and \( \lambda_1 = -1 \).

Applying DT on \( \Psi \) and \( u = x^2 \), we get

$$\Psi[1] = \left( \frac{d}{dx} - x \right) \Psi, \quad u[1] = x^2 - 2,$$

which satisfy the equation

$$-\Psi[1]_{xx} + x^2 \Psi[1] = (\lambda + 2) \Psi.$$

For example: \( \Psi = \exp \left( -\frac{x^2}{2} \right) \) with \( \lambda = 1 \), gives

$$\Psi[1] = -2x \exp \left( -\frac{x^2}{2} \right).$$

with eigenvalue \( \lambda + 2 = 3 \).

Applying the DT on \( \Psi[1] \), we get

$$\Psi[2] = (4x^2 - 2) \exp \left( -\frac{x^2}{2} \right).$$

with eigenvalue \( \lambda + 2 = 5 \).

Applying the DT on \( \Psi[1] \) \( n \)-times, we get

$$\Psi[n] = -H_n \exp \left( -\frac{x^2}{2} \right).$$
with eigenvalue $\lambda + 2 = 2n + 1$, where

$$H_n = (-1)^n \exp \left( \frac{x^2}{2} \right) \left( \frac{d}{dx} - x \right)^n \exp \left( -\frac{x^2}{2} \right)$$  \hspace{1cm} (11)$$

is the Hermite polynomial of order $n$. 
The KdV equation reads

\[ u_t = 6uu_x - u_{xxx}. \] (12)

Lax introduced the pair of operators (denoted afterwards as the *Lax pair*)

\[ L = -\partial_x^2 + u, \quad A = -4\partial_x^3 + 6u\partial_x + 3u_x, \] (13)
such that the KdV equation can be written as

\[ \partial_t L = [L, A]. \] (14)

The last equation can be considered as the *consistency condition* of the following linear system for an *auxiliary* field \( \Psi \)

\[
\begin{align*}
    L\Psi &= \lambda\Psi \\
    \Psi_t &= A\Psi,
\end{align*}
\] (15)

where the consistency condition is

\[(\partial_t L - [L, A])\Psi = 0, \] (16)

is equivalent to the KdV equation.

The linear system (15) is covariant under the DT. Thus

\[
\begin{align*}
    L[1]\Psi[1] &= \lambda\Psi[1] \\
    \Psi[1]_t &= A[1]\Psi[1],
\end{align*}
\] (17)
and the consistency condition becomes

\[(\partial_t L[1] - [L[1], A[1]])\Psi[1] = 0 , \tag{18}\]

which is equivalent to

\[u[1]_t = 6u[1]u[1]_x - u[1]_{xxx} . \tag{19}\]

This means that \(u[1]\) is a new exact solution of the KdV equation which can be obtained from the known exact solution \(u\) via the DT.
DT for Nonlinear Partial Differential Equations

- The Zakharov-Shabat Method

Consider the linear system

\[ \Psi_t = I\Psi \Lambda + U\Psi, \quad \Psi_x = J\Psi \Lambda + U\Psi, \quad (20) \]

where \( \Lambda, I, \) and \( J \) are constant matrices and \([I, J] = 0\). The consistency condition is

\[ U_t - V_x = [I, J], \quad [I, U] = [J, V]. \quad (21) \]

This linear system is found to be covariant under the following version of the DT

\[ \Psi[1] = \Psi \Lambda - \sigma \Psi, \quad \sigma = \Psi_1 \Lambda_1 \Psi_1^{-1}. \quad (22) \]

Applying this DT on the linear system system, we get

\[ U[1] = U + [J, \Psi_1 \Lambda_1 \Psi_1^{-1}]. \quad (23) \]
DT for Nonlinear Partial Differential Equations

- The Nonlinear Schrödinger (NS) Equation

Consider the linear system

\[
\begin{align*}
\Psi_x &= J\Psi\Lambda + U\Psi, \\
\Psi_t &= 2J\Psi\Lambda^2 + 2U\Psi\Lambda + (JU^2 - JU_x)\Psi,
\end{align*}
\]

where

\[
U = \begin{pmatrix} 0 & iq(x, t) \\ ir(x, t) & 0 \end{pmatrix}, \quad J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},
\]

\[
\Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}, \quad \Psi = \begin{pmatrix} \psi_1 & \psi_2 \\ \phi_1 & \phi_2 \end{pmatrix}.
\]

The consistency condition results in the equation

\[
\begin{align*}
iq_t &= q_{xx} + 2q^2 r, \\
ir_t &= -r_{xx} - 2r^2 q.
\end{align*}
\]

Under the condition \( q = r^* \), the last equation becomes

\[
ir_t + r_{xx} + 2|r|^2 r,
\]

which is a nonlinear Schrödinger equation known also as the Gross-Pitaevskii equation (GP).

Applying the DT on the above linear system, we get

\[
U[1] = U + [J, \Psi_1\Lambda\Psi_1^{-1}],
\]

which leads to

\[
r[1] = r - 2(\lambda_{21} - \lambda_{11})\phi_1\phi_2(\psi_1\phi_2 - \phi_1\psi_2)^{-1}, \quad \Lambda_1 = \text{diag}[\lambda_{11}, \lambda_{21}].
\]
Search for the Lax Pairs

- Liang et al. [Phys. Rev. Lett. 95, 050402 (2005).] found the Lax pair for the following GP equation

\[ i \frac{\partial \psi(x,t)}{\partial t} + \frac{\partial^2 \psi(x,t)}{\partial x^2} + \frac{1}{4} \lambda^2 x^2 \psi(x,t) + 2a_0 e^{\lambda t} |\psi(x,t)|^2 \psi(x,t) = 0. \tag{30} \]


\[ i \frac{\partial \psi(x,t)}{\partial t} + K_1(x,t) \frac{\partial^2 \psi(x,t)}{\partial x^2} + K_2(x,t) \psi(x,t) + K_3(x,t) |\psi(x,t)|^2 \psi(x,t) = 0, \tag{31} \]

where \( K_1(x,t), K_2(x,t), \) and \( K_3(x,t) \) are, in principle, arbitrary functions corresponding to effective mass, external potential including loss or gain, and nonlinearity, respectively.

- It turned out that Lax pairs exist only if certain relations between the coefficients \( K_1, K_2, K_3 \) are satisfied [V.N. Serkin et al.,IEEE 8, No.3(2002).].

- As an explicit example, we found the Lax pair of the GP equation

\[ i \frac{\partial \psi(x,t)}{\partial t} + \frac{\partial^2 \psi(x,t)}{\partial x^2} + \frac{1}{4} \lambda(x)^2 \psi(x,t) + 2a(t) |\psi(x,t)|^2 \psi(x,t) = 0, \tag{32} \]

where \( a(t) = a_0 e^{\gamma(t)t} \) and \( \lambda(x) \) and \( \gamma(t) \) are assumed to be independent general functions of \( x \) and \( t \), respectively. For the special case of \( \lambda(x) = \lambda x \) and \( \gamma(t) = \lambda \), the expulsive potential case, Eq. (30), is retrieved.
The method is summarized as follows:

1. We write $\psi$ in the form
   \[ \psi(x, t) = e^{-i f(x) - \gamma(t) t/2} Q(x, t). \] (33)

2. We expand $U$ and $V$ in powers of $Q$:
   \[ U = \begin{pmatrix} f_1 + f_2 Q & f_3 + f_4 Q \\ f_5 + f_6 Q^* & f_7 + f_8 Q^* \end{pmatrix}, \] (34)
   \[ V = \begin{pmatrix} g_1 + g_2 Q + g_3 Q_x + g_4 QQ^* & g_5 + g_6 Q + g_7 Q_x + g_8 QQ^* \\ g_9 + g_{10} Q^* + g_{11} Q_x^* + g_{12} QQ^* & g_{13} + g_{14} Q^* + g_{15} Q_x^* + g_{16} QQ^* \end{pmatrix}, \] (35)
   where $f_{1-8}(x, t)$ and $g_{1-16}(x, t)$ are unknown function coefficients.

3. We require the consistency condition $U_t - V_x + [U, V] = 0$ to give rise to the GP equation (Eq.(32)). This results in 24 equations for the 24 coefficients:
   \[ f_2 = f_3 = f_5 = f_8 = g_2 = g_3 = g_5 = g_8 = g_9 = g_{12} = g_{14} = g_{15} = 0, \]
   \[ f_4 = -f_6 = \sqrt{a_0}, \ g_7 = g_{11} = \sqrt{a_0} i, \ g_4 = -g_{16} = a_0 i. \]
   Using these constant values, the equations for the rest of the coefficients simplify to
   \[ g_{10} = -g_6, \] (36)
\[ f_{1t} - g_{1x} = 0, \]  
(37)  
\[ f_{7t} - g_{13x} = 0, \]  
(38)  
\[ g_{10x} + (f_7 - f_1)g_{10} + \sqrt{a_0} (g_{13} - g_1) - \sqrt{a_0} \left[ -i\lambda^2/4 - (\gamma - 2if_x^2 + \gamma t + 2f_{xx})/2 \right] = 0, \]  
(39)  
\[ g_{10x} - (f_7 - f_1)g_{10} - \sqrt{a_0} (g_{13} - g_1) + \sqrt{a_0} \left[ -i\lambda^2/4 + (\gamma + 2if_x^2 + \gamma t + 2f_{xx})/2 \right] = 0, \]  
(40)  
\[ g_{10} + i\sqrt{a_0}(f_1 - f_7) + 2\sqrt{a_0} f_x = 0. \]  
(41)

4. We solve these equations to obtain the following Lax pair:

\[ U = \begin{pmatrix} f_1 & \sqrt{a_0}Q \\ \alpha_1 f_1 & -\sqrt{a_0}Q^* \end{pmatrix}, \]  
(42)  
\[ V = \begin{pmatrix} g_1 + ia_0|Q|^2 & -g_{10}Q + i\sqrt{a_0}Q_x \\ g_{10}Q^* + i\sqrt{a_0}Q_x^* & \alpha_1 g_1 - ia_0|Q|^2 \end{pmatrix}, \]  
(43)

where

\[ f_1(x, t) = \frac{i\eta_2}{4\lambda_2(\alpha_1 - 1)\eta_1}, \]  
(44)  
\[ g_1(x, t) = \frac{i \left[ (c_2^2\xi^4 + c_3^2)\eta_4 - 2c_2c_3\xi^2\eta_5 \right]}{16(\alpha_1 - 1)\lambda_2^2\eta_1^2}, \]  
(45)
\[ g_{10}(x, t) = -\frac{a_0 \eta_6}{4 \lambda_2 \eta_1}, \]  

where \( \eta_1 = c_3 + c_2 \zeta^2 \), \( \eta_2 = -4 \lambda_2 f_x \eta_1 + (\lambda_1 + 2 \lambda_2^2 x) \eta_3 \), \( \eta_3 = c_3 - c_2 \zeta^2 \), \( \eta_4 = \lambda_1^2 - 4 \lambda_0 \lambda_2^2 \), \( \eta_5 = \lambda_1^2 + \lambda_2^2 (4 \lambda_0 + 8 \lambda_1 x + 8 \lambda_2^2 x^2) \), \( \eta_6 = 4 \lambda_2 f_x \eta_1 + (\lambda_1 + 2 \lambda_2^2 x) \eta_2 \), and \( \zeta = \exp(\lambda_2 t) \).

Calculating the consistency condition using this Lax pair, we obtain the Gross-Pitaevskii equation

\[
i \frac{\partial \psi(x, t)}{\partial t} + \frac{\partial^2 \psi(x, t)}{\partial x^2} + \frac{1}{4} (\lambda_0 + \lambda_1 x + \lambda_2^2 x^2) \psi(x, t) + \frac{2a_0}{c_2 e^{\lambda_2 t} + c_3 e^{-\lambda_2 t}} |\psi(x, t)|^2 \psi(x, t) = 0,
\]

where \( \lambda_{0,1,2}, c_2 \) and \( c_3 \) are arbitrary coefficients.

5. For the special case \( c_2 = 1 \) and \( c_3 = 0 \), the above Lax pair and GP equation reduce to those of Liang et al..

6. A seed solution for the last GP equation can be derived for the general case:

\[
\psi(x, t) = A \sqrt{\text{sech} \left[ \lambda_2 (2c_3 + t) \right]} \\
\times \exp \left\{ c_4 + i \left[ c_1(t) + c_2(t)x + \lambda_2 \tanh \left[ \lambda_2 (2c_3 + t) \right] x^2 / 4 \right]\right\},
\]

(48)
where $c_1(t)$ and $c_2(t)$ are given by

\[
\begin{align*}
\frac{\partial p}{\partial t} & = c_6 + \frac{1}{16\lambda_2^3} \left\{ \lambda_2 (c_7 - t)(\lambda_1^2 - 4\lambda_0 \lambda_2^2) 
+ 8c_5 \lambda_1 \lambda_2 (\text{sech } \eta_1 - \text{sech } \eta_2) 
+ (\lambda_1^2 - 16c_5^2 \lambda_2^2)(\tanh \eta_1 - \tanh \eta_2) 
+ 2A^2 e^{2c_4} g_0 \int_{c_7}^t dt' \gamma(t') \text{sech } [\lambda_2 (2c_3 + t')] \right\}, \\
c_2(t) &= c_5 \text{sech } \eta_1 + \frac{\lambda_1}{4\lambda_2} \tanh \eta_1,
\end{align*}
\]

where $\eta_1 = \lambda_2(2c_3 + t)$, $\eta_2 = \lambda_2(2c_3 + c_7)$, and $c_3-7$ are constants of integration.
Exact Solitonic Solutions of the Gross-Pitaevskii Equation

- Linear Inhomogeneity

Using the previously-described method, we found the Lax representation for the equation

\[ i \frac{\partial \psi(x, t)}{\partial t} = \left[ -\frac{\partial^2}{\partial x^2} + x - p^2 |\psi(x, t)|^2 \right] \psi(x, t), \quad (51) \]

namely

\[ \Psi_x = J \Psi \Lambda + U \Psi, \quad (52) \]

\[ i \Psi_t = W \Psi + 2(\zeta J + U) \Psi \Lambda + 2 J \Psi \Lambda^2, \quad (53) \]

where,

\[ \Psi(x, t) = \begin{pmatrix} \psi_1(x, t) & \psi_2(x, t) \\ \phi_1(x, t) & \phi_2(x, t) \end{pmatrix}, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (54) \]

\[ \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad U = \begin{pmatrix} \zeta & pq(x, t)/\sqrt{2} \\ -pr(x, t)/\sqrt{2} & -\zeta \end{pmatrix}, \quad (55) \]

\[ W = (\zeta^2 - x/2) J + 2 \zeta U - J (U^2 - U_x), \quad (56) \]

\[ \zeta(t) = it/2, \text{ and } \lambda_1 \text{ and } \lambda_2 \text{ are arbitrary constants and} \]
\( q(x, t) = r^*(x, t) = \psi(x, t) \), and \( p = 1 \) for attractive interactions and \( p = \pm i \) for repulsive interactions.

Applying the DT, using the seed \( \psi_0(x, t) = A \exp(i\phi_0) \) we obtain for the solution of the repulsive interactions case \( (p = \pm i) \)

\[
\psi(x, t) = e^{i\phi_0} \left[ A \pm i\sqrt{8}\lambda_1 r \times \frac{2u_r^+ \cosh \theta - 2iu_i^+ \sinh \theta + (|u^+|^2 + 1) \cos \beta + i(|u^+|^2 - 1) \sin \beta}{(|u^+|^2 - 1) \sinh \theta + 2u_i^+ \sin \beta} \right] (57)
\]

and for the attractive interactions case \( (p = 1) \), the solution is

\[
\psi(x, t) = e^{i\phi_0} \left[ A - \sqrt{8}\lambda_1 r \times \frac{2u_r^+ \cosh \theta - 2iu_i^+ \sinh \theta + (|u^+|^2 + 1) \cos \beta + i(|u^+|^2 - 1) \sin \beta}{(|u^+|^2 + 1) \cosh \theta + 2u_r^+ \cos \beta} \right] (58)
\]

where \( \phi_0 = t(p^2A^2 - (t^2/3 + x)) \),
\[
\theta = \sqrt{2} \left[ \Delta_r(t^2 + x) + 2(\Delta_r \lambda_{1i} - \Delta_i \lambda_{1r})t \right] - \delta_r,
\]
\[
\beta = -\sqrt{2} \left[ \Delta_i(t^2 + x) + 2(\Delta_i \lambda_{1i} + \Delta_r \lambda_{1r})t \right] + \delta_i, u^\pm = \sqrt{8} p A/b^\pm,
\]
\[
b^\pm = 4\lambda_i^* \pm \Delta, \Delta = \sqrt{2\lambda_i^*^2 - p^2 A^2}, \text{ and } \delta \text{ is an arbitrary constant.}
Exact Solitonic Solutions of the Gross-Pitaevskiiii Equation

- Linear Inhomogeneity: Properties of the Solution

Depending on the arbitrary constants, we have three types of solutions:

i) oscillatory solutions.

ii) nonoscillatory solutions, i.e., single-soliton solution.

iii) oscillatory with a localized envelope.

![Graphs showing density](image)

Figure 1: Density $\rho(x) = |\psi(x)|^2$ at time $t = 0$ for the case of attractive interactions. The arbitrary constants chosen to generate these plots are: $\delta_r = \delta_i = 0$ and $A = 1$ for all plots. In (a) $\lambda_{1i} = 0, \lambda_{1r} = 0.29$, in (b): $\lambda_{1i} = 0, \lambda_{1r} = -0.6$, in (c): $\lambda_{1i} = 0, \lambda_{1r} = 0.8$, in (d): $\lambda_{1i} = 0, \lambda_{1r} = -1.5$, in (e): $\lambda_{1i} = 2, \lambda_{1r} = -0.6$, and in (f): $\lambda_{1i} = 2, \lambda_{1r} = 0.29$. 
Exact Solitonic Solutions of the Gross-Pitaevskii Equation

- Linear Inhomogeneity: Properties of the Solution

Time evolution of solitons:
i) for single solitons: \( x = -t^2 - 2(\lambda_1 - \lambda_1 \Delta_i / \Delta_r)t + \delta_r / \sqrt{2} \)

ii) for multiple solitons: \( x = -t^2 - 2(\lambda_1 + \lambda_1 \Delta_r / \Delta_i)t + \delta_i / \sqrt{2} \).

→ Parabolic with trajectory acceleration of -1.

In real units, this acceleration equals \( -F/m \), where \( F \) is the force constant of the linear potential and \( m \) is the mass of an atom.

Figure 2: Surface plots of the density \( \rho(x, t) = |\psi(x, t)|^2 \) versus \( x \) and \( t \) for the attractive interactions. The upper plot corresponds to Fig. 1(d) while the lower figure corresponds to Fig 1(a).
Exact Solitonic Solutions of the Gross-Pitaevskii Equation

- Linear Inhomogeneity: Properties of the Solution

The peak of the soliton is maximum at times:

\[ t = n\pi \Delta_r / \sqrt{8|\Delta|^2\lambda_1}, \] where \( n \) is an integer.

![Graph](image)

Figure 3: The soliton density along its trajectory for attractive interactions. The upper figure corresponds to Fig. 1(c) and the lower figure corresponds to Fig. 1(d).
Exact Solitonic Solutions of the Gross-Pitaevskii Equation

- Linear Inhomogeneity: Properties of the Solution

Phase of Solitons:
Phase difference equals $\pi$ and is a constant of time.

Figure 4: The density (solid curve) and phase (light curve) of a solitonic solution for attractive interactions with $\delta_r = \delta_i = 0$, $A = 1$, $\lambda_{1i} = 0.03$, and $\lambda_{1r} = 0.6$. The upper plot is for $t = 0$ and the lower plot is for $t = 0.85\tau$. 
Search is still ongoing for more exact solutions...